Barnett Relaxation in Thermally-Rotating Grains

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ABSTRACT

We present an exact formulation of the physics of Barnett relaxation. Our formulation is based on a realistic kinetic model of the relaxation mechanism which includes the alignment of the grain angular momentum in body coordinates by Barnett dissipation, disalignment by thermal fluctuations, and coupling of the angular momentum to the gas via gas damping. We solve the Fokker-Planck equation for the measure of internal alignment using numerical integration of the equivalent Langevin equation for Brownian rotation. The accuracy of our results is calibrated by comparing our numerical solutions with exact analytic results obtained for special cases. We describe an analytic approximation for the measure of alignment which fits our numerical results for cases of practical interest.

Subject headings: dust, extinction — ISM, clouds — ISM, polarization

1. Introduction

The polarization of starlight, first discovered by Hiltner (1949) and Hall (1949), implies that the axes of dust grains are partially aligned with the interstellar magnetic field, $\boldsymbol{B}$. Models of the Davis-Greenstein (DG) effect and other grain alignment mechanisms generally predict the alignment of the grain angular momentum, $\boldsymbol{J}$, with respect to $\boldsymbol{B}$ (for a recent review, see Roberge 1996). However, to relate the predictions of these models to observations, one also requires the "internal distribution" of $\boldsymbol{J}$ in grain body coordinates. The classical papers on DG alignment (Jones & Spitzer 1967; Purcell & Spitzer 1971) assumed that the internal distribution is determined by gas damping and other external processes.[1] However,

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[1]For example, Jones & Spitzer (1967) noted that the component of $\boldsymbol{B}$ parallel to the grain angular velocity can enhance the alignment of $\boldsymbol{J}$ in grain coordinates.

Purcell (1979, hereafter P79) realized that frictional processes internal to the grain will dissipate rotational energy into heat on a timescale that is generally much shorter than the timescales for external interactions. Since they conserve angular momentum, these internal dissipation processes tend to drive the grain toward the state of minimum energy consistent with a fixed value of $\boldsymbol{J}$, rotation about the principal axis of largest rotational inertia (henceforth the "major axis of inertia").

Several internal dissipation processes were discussed by P79, who found that "Barnett dissipation" is the dominant effect for typical interstellar grains. The mechanism is very elegant: Dolginov & Mytrophanov (1976) pointed out that a rotating grain has a magnetization $\boldsymbol{M} = -\chi\boldsymbol{\Omega}/\gamma$ due to the Barnett effect, where χ is the magnetic susceptibility, γ is the magnetogyric ratio of the paramagnetic spins, and $\boldsymbol{\Omega}$ is the grain angular velocity. It is well known from theoretical mechanics (e.g., Landau & Lifshitz 1976) that $\boldsymbol{\Omega}$ precesses in body coordinates unless the rotation axis coincides with a principal axis of the inertia tensor. Reasoning by analogy with ordinary paramagnetic dissipation in a time-dependent external magnetic field, P79 concluded that internal dissipation must also be caused by the time-dependent "Barnett equivalent magnetic field," $\boldsymbol{H}_{\mathrm{Be}} = -\boldsymbol{\Omega}/\gamma$.

Virtually all models of interstellar polarization developed since 1979 have assumed that Barnett dissipation aligns $\boldsymbol{J}$ *perfectly* with the major axis of inertia. However, Lazarian (1994, henceforth L94) has pointed out that this is a poor approximation if the grains rotate with thermal kinetic energies: thermal fluctuations in the Barnett magnetization will excite rotation about all 3 of the body axes, preventing perfect alignment unless the rotation is suprathermal or the grain solid temperature is zero.

The first model of internal relaxation to include thermal fluctuations was presented by L94, who modeled the kinetics of Barnett relaxation by analogy with the DG effect. However, this analogy is not exact because, unlike Barnett relaxation, the DG effect involves external, nonconservative torques. This difficulty was ignored for the sake of simplicity by L94 and we now revisit the problem in order to carry out an exact analysis. In §2 we compare the efficiency of various dissipation mechanisms in order to delineate the circumstances under which the Barnett effect is the dominant process. In §3 we discuss Barnett alignment in an isolated grain whose angular momentum has a fixed but arbitrary magnitude, J. The value of J is determined by interactions of the grain with its environment and in §4 we formulate a kinetic model wherein the external interactions are provided by gas-grain collisions and the evaporation of molecules from the grain surface. We calculate the appropriate measure of internal alignment numerically in §5 and calibrate the accuracy of our results by considering special cases where exact solutions can be obtained analytically. Our results are discussed in §6 and summarized in §7.

2. Internal Dissipation

Inelastic dissipation and Barnett dissipation were identified in P79 as the two major causes of internal dissipation. Inelastic dissipation is caused by the oscillating stresses associated with the motion of the grain angular velocity, $\boldsymbol{\Omega}$, in body coordinates. It is possible to show that the timescale for inelastic dissipation, t_{ai}, scales as Ω^{-3} compared to Ω^{-2} for the Barnett dissipation time, t_{Bar} (see Eq. (2-4)). As a result, inelastic dissipation is the dominant process for sufficiently large angular velocities. However, for sufficiently rigid grains, these velocities lie in the range of suprathermal rotation; therefore, we consider only Barnett dissipation in the present paper, which deals with thermally rotating grains. The interesting problem of inelastic dissipation in dust grains will be discussed elsewhere.

As we pointed out above, Barnett dissipation tends to align $\boldsymbol{J}$ with the major axis of inertia (see §1). Henceforth we will model the grains as oblate spheroids with semiaxes a parallel to $\hat{\boldsymbol{z}}$ and b (with $b > a$) perpendicular to $\hat{\boldsymbol{z}}$, where $\hat{\boldsymbol{z}}$ is the symmetry axis (=the major axis of inertia). Barnett dissipation decreases the angle θ between $\boldsymbol{J}$ and $\hat{\boldsymbol{z}}$ at a rate

$$\frac{d\theta}{dt} = -G_{\mathrm{Bar}} \sin\theta \cos\theta \tag{2-1}$$

(P79; Roberge, DeGraff, & Flaherty 1993, henceforth RDGF93), where

$$G_{\mathrm{Bar}} = \frac{VKh^2(h-1)J^2}{\gamma^2 I_z^3}, \tag{2-2}$$

V is the grain volume, I_x is the inertia for rotation about an axis perpendicular to $\hat{\boldsymbol{z}}$, $h \equiv I_z/I_x$, K is related to the imaginary part of the magnetic susceptibility at frequency Ω by

$$K \equiv \frac{\mathrm{Im}[\chi(\Omega)]}{\Omega} \approx 1.2 \times 10^{-13} \left(\frac{T_s}{15K}\right), \tag{2-3}$$

and T_{s} is the grain solid temperature. The scaling of K in equation (2-3) is appropriate for ordinary paramagnetic materials (see Draine 1996); for superparamagnetic grains, K should be increased by an uncertain factor $\sim 10^5$. We define the Barnett time, t_{Bar}, to be the value of G_{Bar}^{-1}, so that

$$t_{\mathrm{Bar}}(J) = \frac{\gamma^2 I_z^3}{VKh^2(h-1)J^2}. \tag{2-4}$$

The numerical value of t_{Bar} depends on J; a typical angular momentum for thermal rotation at the gas temperature, T_{g}, is

$$J_{\mathrm{therm}} = \sqrt{I_z k T_{\mathrm{g}}} \tag{2-5}$$

and the corresponding numerical value of t_{Bar} is

$$t_{\mathrm{Bar}}(J_{\mathrm{therm}}) = 4 \times 10^6 \rho_{s0}^2 \, T_{g1}^{-1} \, K_{-13}^{-1} \, b_{-5}^7 \, \frac{(a/b)\left(1 + a^2/b^2\right)^3}{(1 - a^2/b^2)} \quad \mathrm{s}, \tag{2-6}$$

for a grain with radius $10^{-5}b_{-5}$ cm and density ρ_{s0} g cm^{-3}. We may say that eq. (2-4) describes Barnett dissipation for an individual grain with fixed angular momentum J, while eq. (2-6) decsribes Barnett dissipation for an ensemble of grains with rotational temperature T_g.

3. Barnett Dissipation in an Isolated Grain

Barnett dissipation is able to transform rotational energy into heat because coupling exists between the grain rotational and vibrational degrees of freedom. However, the existence of this coupling implies that energy can also be transferred in the opposite direction, i.e., from vibration to rotation (see Jones & Spitzer 1967). As a result, fluctuations in the thermal energy will randomly excite rotation about all three body axes with energies of order kT_s. If the rotational energy of the grain is $\gg kT_s$, then these thermal fluctuations are negligible[2] and Barnett dissipation aligns $\boldsymbol{J}$ perfectly with $\hat{\boldsymbol{z}}$. This is the assumption adopted by P79, who was concerned only with rotation at highly superthermal energies. However, the grains in molecular clouds are likely to have rotational energies $\sim kT_s$. Consequently, it is of great practical interest to determine the distribution of orientations established by the combined effects of Barnett dissipation and thermal fluctuations.

This distribution can be calculated analytically for an isolated grain. Let $f_{TE}(\theta \,|\, J)$ be the distribution function for θ, defined so that $f_{TE} \sin\theta \, d\theta$ is the probability of finding $\boldsymbol{J}$ oriented between θ and $\theta + d\theta$. Here we regard f_{TE} to be a function of θ with J a fixed but arbitrary parameter; the determination of J by external interactions is discussed in §4. Since the grain is assumed to be an isolated system in thermodynamic equilibrium, f_{TE} can be found, in principle, by maximizing the entropy of the combined rotation/vibration system. However, the calculation is drastically simplified by noting that the number of vibrational degrees of freedom ($\sim N$, where N is the number of atoms in the grain) is vastly greater than the number of rotational degrees of freedom (=3). Thus, the rotational degrees of freedom may be regarded as a small system immersed in a heat bath at constant temperature, T_s.[3]

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[2]However, see Lazarian & Draine 1997, where it is shown that the effects of incomplete Barnett alignment are important during "crossovers" in suprathermally rotating grains.

[3]Here we assume for simplicity that T_s is maintained at a constant value by the various processes (interactions with the ambient radiation field, gas damping, etc) by which energy is exchanged between the grain and its environment. That is, we assume that the radiation field, gas temperature, etc change on a timescale that is much longer than t_{Bar}.

It follows that f_{TE} is just the Boltzmann distribution,

$$f_{\mathrm{TE}}\left(\theta\,|\,J\right) = C\,\exp[-E_{\mathrm{rot}}(\theta)/kT_{\mathrm{s}}], \tag{3-1}$$

where C is a normalization constant and

$$E_{\mathrm{rot}}(\theta) = \frac{J^2}{2I_z}\left[1 + (h-1)\sin^2\theta\right] \tag{3-2}$$

is the rotational energy.

In a steady state, the linear and circular polarization produced by an ensemble of spheroidal grains depend only on the measure of internal alignment,

$$Q_X \equiv \frac{3}{2}\left[\left\langle\cos^2\theta\right\rangle - \frac{1}{3}\right], \tag{3-3}$$

where angle brackets denote the distribution average. Evaluating the average for f_{TE}, we find that the measure of internal alignment for an isolated grain is

$$Q_{X,\mathrm{TE}}\left(a/b, T_{\mathrm{s}}, J\right) = \frac{3}{2\xi^2}\left[\frac{\int_0^\xi t^2\,\exp\left(t^2 - \xi^2\right)\,dt}{\int_0^\xi \exp\left(t^2 - \xi^2\right)\,dt}\right] - \frac{1}{2}, \tag{3-4}$$

where we have introduced the dimensionless "suprathermality parameter,"

$$\xi^2 \equiv \frac{(h-1)\,J^2}{2I_z kT_{\mathrm{s}}}. \tag{3-5}$$

The dependence of $Q_{X,\mathrm{TE}}$ on ξ (Fig. 1) has a straightforward physical interpretation. For a fixed value of J, the difference in rotational energy between the minimum ($\theta = 0$) and maximum ($\theta = \pi/2$) rotational energy "states" of a grain is $\Delta E = (h-1)\,J^2/2I_z$ [see eq. (3-2)]. Thus, ξ^2 is just the ratio of the energy required to disorient $\boldsymbol{J}$ to the energy added to rotation by a typical thermal fluctuation. Weak alignment ($\xi \lesssim 1$) occurs when a typical fluctuation is large enough to disorient $\boldsymbol{J}$ ($kT_{\mathrm{s}} \gtrsim \Delta E$), corresponding to nearly-spherical grains ($h \approx 1$) or thermal rotation energies ($J^2/2I_z \sim kT_{\mathrm{s}}$). Strong alignment ($\xi \gg 1$) occurs only for nonspherical grains and suprathermal rotation.

4. Barnett Dissipation with External Interactions

4.1. Mathematical Description

The effects of external interactions can be included by solving the Fokker-Planck equation,

$$\frac{\partial f}{\partial t} = -\boldsymbol{\nabla}\cdot\boldsymbol{S}, \tag{4-1}$$

where t is time, $f(J, \theta)$ is the joint distribution for J and θ, and $\boldsymbol{S}$ is the probability current. The latter is given in general by

$$\boldsymbol{S} = \boldsymbol{A} f - \frac{1}{2} \boldsymbol{\nabla} \cdot (B f), \qquad (4\text{-}2)$$

where

$$\boldsymbol{A} \equiv \left\langle \frac{\Delta \boldsymbol{J}}{\Delta t} \right\rangle \qquad (4\text{-}3)$$

is the mean torque and

$$B \equiv \left\langle \frac{\Delta \boldsymbol{J} \Delta \boldsymbol{J}}{\Delta t} \right\rangle \qquad (4\text{-}4)$$

is the diffusion tensor. In principle, $\boldsymbol{A}$ and B include the cumulative effects of every process that changes $\boldsymbol{J}$ in the body frame. In this paper, we consider the effects of Barnett dissipation and gas damping but neglect the effects of other external interactions such as the Davis-Greenstein mechanism. As a result, our calculations accurately describe the internal angular momentum distribution only under conditions where the gas damping time is much smaller than the timescale for other external processes.[4] We assume that the grain center of mass is at rest with respect to the gas; the generalization of the results presented here to Gold-type alignment will be considered in subsequent papers (Lazarian 1997b; Roberge & Lazarian 1997b). We also assume that the properties of the grains and their environment are steady on timescales much longer than the gas damping and Barnett times, so that only the steady solution of the Fokker-Planck equation is of interest.

4.2. Diffusion Coefficients for Barnett Relaxation

It is simplest to calculate $\boldsymbol{A}^{(\mathrm{Bar})}$ and $B^{(\mathrm{Bar})}$, the mean torque and diffusion tensor associated with Barnett relaxation, in a spherical polar coordinate system with basis $\left(\hat{\boldsymbol{J}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}} \right)$, where the polar axis is parallel to the grain symmetry axis. The conservation of J by the Barnett effect and the azimuthal symmetry of an oblate spheroid together imply that the only nonvanishing components are $A_\theta^{(\mathrm{Bar})}$ and $B_{\theta\theta}^{(\mathrm{Bar})}$. The former follows immediately from equation (2-1) and is given by

$$A_\theta^{(\mathrm{Bar})} = -\frac{J}{t_{\mathrm{Bar}}} \sin\theta \cos\theta \qquad (4\text{-}5)$$

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[4]In particular, we do not attempt to describe magnetic dissipation by ordinary or superparamagnetic grains. These effects will be considered in subsequent papers on Davis-Greenstein alignment (Lazarian 1997a; Roberge & Lazarian 1997a), which generalize the treatments of DG alignment given in Roberge et al. (1993) and Lazarian (1995a).

where t_{Bar} is defined by eq. (2-4). The analogous expression for $B_{\theta\theta}^{(\mathrm{Bar})}$ can be determined by noting that, in the absence of other processes, Barnett relaxation must establish the thermodynamic equilibrium distribution [eq. (3-1)]. Further, the rate of every microscopic process equals the rate of the time-reversed process in thermodynamic equilibrium (the "principle of detailed balance"), so the probability current must vanish at every point in phase space. Writing out the components of $\boldsymbol{S}$ in polar coordinates and setting $S_\theta = 0$, we find that detailed balance requires

$$\sin\theta\, f_{\mathrm{TE}}\, A_\theta^{(\mathrm{Bar})} - \frac{1}{2J}\frac{\partial}{\partial\theta}\left[\sin\theta\, f_{\mathrm{TE}}\, B_{\theta\theta}^{(\mathrm{Bar})}\right] = 0. \tag{4-6}$$

Expression (4-6) is a first-order differential equation for $B_{\theta\theta}^{(\mathrm{Bar})}$. The solution is

$$B_{\theta\theta}^{(\mathrm{Bar})} = \frac{J^2}{t_{\mathrm{Bar}}\sin\theta}\left[\exp(\xi^2\sin^2\theta)\int_{\sin^2\theta}^{1}\sqrt{y}\exp(-\xi^2 y)\mathrm{d}y + \exp(-\xi^2\cos^2\theta)\right]. \tag{4-7}$$

In solving equation (4-6), we have imposed the condition[5]

$$B_{\theta\theta}^{(\mathrm{Bar})}(\frac{\pi}{2},\xi) = \frac{J^2}{t_{\mathrm{Bar}}\xi}, \tag{4-8}$$

which insures that the $B_{\theta\theta}^{(\mathrm{Bar})}$ is smooth at $\pi/2$:

$$\lim_{\xi\to\infty}\frac{\mathrm{d}^2 B_{\theta\theta}^{(\mathrm{Bar})}(\pi/2,\xi)}{\mathrm{d}\theta^2} < \infty. \tag{4-9}$$

When $J^2 \ll I_z kT_{\mathrm{s}}$ thermal fluctuations easily change its direction and this causes $B_{\theta\theta}^{(\mathrm{Bar})} \to \infty$ in agreement with our results.

4.3. Diffusion Coefficients for Gas Damping

The diffusion coefficients for gas damping were calculated by RDGF93 for oblate spheroids on the assumption that $\boldsymbol{\Omega}$ is aligned perfectly with $\hat{\boldsymbol{z}}$ (i.e., perfect Barnett alignment). However, the diffusion tensor is independent of $\boldsymbol{\Omega}$ to first order in $\Omega b/v_{\mathrm{th}}$, where $v_{\mathrm{th}} = \sqrt{2kT_{\mathrm{g}}/m}$ is the gas thermal speed. We may therefore adopt the results of RDGF93 for the diffusion tensor with high accuracy. The nonzero Cartesian components are

$$B_{xx}^{(\mathrm{gas})} = \frac{2\sqrt{\pi}}{3}nmb^4 v_{\mathrm{th}}^3\Gamma_\perp(e)\left(1+\frac{T_{\mathrm{s}}}{T_{\mathrm{g}}}\right), \tag{4-10}$$

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[5]It worth noting that condition (4-8) coincides with the result for the diffusion coefficient at a critical point obtained in Zeldovich (1942) for the problem of phase transitions of the first order.

$$B_{yy}^{(\text{gas})} = B_{xx}^{(\text{gas})}, \tag{4-11}$$

and

$$B_{zz}^{(\text{gas})} = \frac{2\sqrt{\pi}}{3} nmb^4 v_{\text{th}}^3 \Gamma_\parallel(e) \left(1 + \frac{T_\text{s}}{T_\text{g}}\right), \tag{4-12}$$

where n is the gas number density and the geometrical factors $\Gamma_\parallel$ and $\Gamma_\perp$ are dimensionless functions of a/b. They are given in terms of the eccentricity, $e = \sqrt{1 - a^2/b^2}$, by

$$\Gamma_\parallel(e) = \frac{3}{16} \left\{ 3 + 4(1 - e^2)g(e) - e^{-2} \left[1 - (1 - e^2)^2 g(e)\right]\right\} \tag{4-13}$$

and

$$\Gamma_\perp(e) = \frac{3}{32} \left\{ 7 - e^2 + (1 - e^2)^2 g(e) + (1 - 2e^2) \left[1 + e^{-2}\left[1 - (1 - e^2)^2 g(e)\right]\right]\right\}, \tag{4-14}$$

respectively, where

$$g(e) \equiv \frac{1}{2e} \ln\left(\frac{1 + e}{1 - e}\right). \tag{4-15}$$

The mean torque due to gas damping vanishes to zeroth order in Ω, so we cannot adopt the expressions given by RDGF93. However, one can relate the mean torque to the diffusion tensor via the principle of detailed balance[6] with the result

$$A_x^{(\text{gas})} = -\frac{4\sqrt{\pi}}{3I_x} nmb^4 v_{\text{th}} \Gamma_\perp J_x, \tag{4-16}$$

$$A_y^{(\text{gas})} = -\frac{4\sqrt{\pi}}{3I_x} nmb^4 v_{\text{th}} \Gamma_\perp J_y, \tag{4-17}$$

and

$$A_z^{(\text{gas})} = -\frac{4\sqrt{\pi}}{3I_z} nmb^4 v_{\text{th}} \Gamma_\parallel J_z. \tag{4-18}$$

If the Barnett alignment is perfect, then $J_x = J_y = 0$ and the preceding expressions reduce to the components obtained in RDGF93. The characteristic timescale for gas damping is

$$t_{\text{gas}} = \frac{3I_z}{4\sqrt{\pi}nmb^4 v_{\text{th}}\Gamma_\parallel}, \tag{4-19}$$

where

$$t_{\text{gas}} = 7 \times 10^9 \, \rho_{\text{s0}} \, n_4^{-1} \, T_{\text{g1}}^{-1/2} \, a_{-5} \Gamma_\parallel \quad \text{s} \tag{4-20}$$

for a gas of pure H_2.

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[6]Strictly speaking, this principle applies only in thermodynamic equilibrium with $T_\text{s} = T_\text{g}$. However, it can be shown that the mean torque is independent of T_s/T_g to first order in $\Omega b/v_{\text{th}}$ so that, apart from a small error, the thermodynamic equilibrium result equals the result for $T_\text{s} \neq T_\text{g}$.

4.4.   Numerical Methods

We solve the Fokker-Planck equation indirectly by integrating the equivalent "Langevin equation for Brownian rotation,"

$$d\boldsymbol{J} = \boldsymbol{A}\, dt + \sqrt{B}\, d\boldsymbol{W} \tag{4-21}$$

(RDGF93), where $\sqrt{B}$ denotes the matrix square root of B and $d\boldsymbol{W}$ is a vector of statistically independent, Gaussian random variables with variance dt. To calculate Q_X, we integrate equation (4-21) numerically to generate a simulation of $\boldsymbol{J}(t)$ and equate the ensemble average of $\cos^2 \theta$ in eq. (3-3) to the time average over the history of the simulation. The numerical integration was performed using the Euler algorithm; this scheme approximates the change in J_i during each time step, Δt, by

$$\Delta J_i = A_i\, \Delta t + \left(\sqrt{B}\right)_{ij} \Delta W_j, \tag{4-22}$$

where the ΔW_j are independent samples from a Gaussian distribution with variance $(\Delta t)^2$. The numerical integration was performed using a "mixed" scheme, where the changes in J_x, J_y, and J_z due to gas damping were computed from the Euler algorithm in Cartesian components and the change in θ due to Barnett relaxation was computed from the Euler algorithm in polar coordinates.

5.   Results

5.1.   Dimensionless Parameters

The number of independent parameters in the problem is greatly reduced by rewriting the Fokker-Planck equation in terms of dimensionless time and angular momentum variables. With time measured in units of t_{Bar} and angular momentum measured in units of J_{therm}, only 3 dimensionless parameters appear:

$$a/b \equiv \text{grain axis ratio}, \tag{5-1}$$

$$T_{\mathrm{s}}/T_{\mathrm{g}} \equiv \frac{\text{dust solid temperature}}{\text{gas kinetic temperature}}, \tag{5-2}$$

and

$$\delta_B \equiv \frac{t_{\mathrm{gas}}}{t_{\mathrm{Bar}}(J_{\mathrm{therm}})}. \tag{5-3}$$

Henceforth we will refer to δ_B as the "Barnett damping parameter." In the model adopted here, where gas damping is the only external interaction, Q_X is completely determined by specifying the values of a/b, $T_{\mathrm{s}}/T_{\mathrm{g}}$, and δ_{B}.

5.2. Benchmark Tests

The accuracy of our numerical method can be calibrated by considering two special cases where exact analytic solutions for Q_X exist. Consider first the case of an isolated grain, for which $Q_X = Q_{X,\mathrm{TE}}$. To simulate an isolated grain, we assigned J/J_{therm} as an initial condition, turned off gas damping in our numerical code, and integrated the Langevin equation with the timestep and averaging time set to $\Delta t = 1 \times 10^{-3}$ and $T = 2 \times 10^4$ dimensionless units, respectively. The measures of alignment computed for several cases are plotted as solid circles in Figure 1. The rms error in Q_X is 4×10^{-3} for the numerical solutions in Figure 1.

Exact solutions can also be obtained in the "Maxwellian limit," $\delta_{\mathrm{B}} = 0$, where the timescale for Barnett relaxation is infinitely long compared to the timescale for gas damping. The Maxwellian distribution for θ is

$$f_{\mathrm{Max}}(\theta) = \frac{h}{4\pi} \left(\cos^2 \theta + h \sin^2 \theta \right)^{-3/2} \tag{5-4}$$

(Jones & Spitzer 1967, but note that we have corrected a minor typographical error) and the corresponding measure of alignment is

$$Q_{X,\mathrm{Max}} = \frac{3}{2\left(1 - h^{-1}\right)} \left[1 - \frac{1}{\sqrt{h-1}} \sin^{-1}\left(1 - h^{-1}\right) \right] - \frac{1}{2}. \tag{5-5}$$

In the Maxwellian regime, the partial alignment of $\boldsymbol{J}$ with $\hat{\boldsymbol{z}}$ is due entirely to the "inertial asphericity" of the grains and, appropriately, Q_X depends only on a/b.[7] In Figure 2, we compare the exact solution for $Q_{X,\mathrm{Max}}$ (solid curve) with numerical solutions calculated for a few values of a/b (filled circles). The numerical solutions were obtained by integrating the Langevin equation as a function of time in units of t_{gas} (since t_{Bar} is irrelevant for this case), with $\Delta t = 1 \times 10^{-3}$ and $T = 2 \times 10^4$ dimensionless units, respectively. The rms error in Q_X for the filled circles in Figure 2 is 2×10^{-3}.

5.3. Barnett Relaxation with Gas Damping

Typical results for realistic calculations including gas damping and Barnett relaxation are shown in Figures 3 and 4. As noted above, the case $\delta_B = 0$ corresponds to the Maxwellian regime and, appropriately, all of the solutions in Figures 3 and 4 converge to the Maxwellian case as $\delta_B \to 0$. As δ_B increases, Barnett relaxation becomes increasingly important and

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[7]Recall that $h = 2/\left(1 + a^2/b^2\right)$ for a homogeneous, oblate spheroid.

Q_X diverges from $Q_{X,\mathrm{Max}}$. Notice that Barnett relaxation always *enhances* the alignment, in the sense that $Q_X > Q_{X,\mathrm{Max}}$ for $\delta_B > 0$. For large δ_B, the numerical solution for each combination of a/b and $T_\mathrm{s}/T_\mathrm{g}$ approaches a limiting value

$$Q_{X,\mathrm{inf}}\left(a/b, T_\mathrm{s}/T_\mathrm{g}\right) \equiv \lim_{\delta_B \to \infty} Q_X\left(a/b, T_\mathrm{s}/T_\mathrm{g}, \delta_B\right), \qquad (5\text{-}6)$$

which depends only on the grain shape and dust-to-gas temperature ratio. This limit represents the maximum possible enhancement of the alignment (relative to $Q_{X,\mathrm{Max}}$) which can be produced by Barnett relaxation in a thermally-rotating grain. The qualitative behavior of the results presented in Figures 3 and 4 is consistent with the predictions of L94. For example, the maximum efficiency of Barnett alignment decreases as $T_\mathrm{s}/T_\mathrm{g}$ increases and as the grains become more spherical.

Note that the oscillations which occur for large δ_B in Figs. 3 and 4 are spurious, that is, they are too large to be statistical fluctuations. The oscillations are probably an artifact of the ad hoc prescription we use to treat the singularities in $B_{\theta\theta}^{(\mathrm{Bar})}$ at $\theta = 0$ and $\theta = \pi$ [see eq. (4-7)]: in computing $B_{\theta\theta}^{(\mathrm{Bar})}$, we set $\theta = \theta_{\min}$ whenever the trajectory "visits" the interval $(0, \theta_{\min})$ and similarly for points in the interval $(\pi - \theta_{\min}, \pi)$. This procedure is motivated by the fact that the fluctuations in θ are proportional to $\sqrt{B_{\theta\theta}^{(\mathrm{Bar})}}$ [see eq. (4-21)]. For some value of $\theta_{\min}$, the fluctuations for $\theta < \theta_{\min}$ become so large that the grain is completely disoriented and, roughly speaking, the precise values of the fluctuations ought to be irrelevant. After experimenting with different choices, we set $\theta_{\min} = 0.1\,\mathrm{rad}$ for the calculations presented in Figs. 3 and 4.

The fact that the solutions in Figures 3 and 4 approach limiting values for large δ_B is a consequence of elementary thermodynamic considerations. The case $\delta_B \gg 1$ corresponds to conditions where θ evolves rapidly (on a timescale $\sim t_{\mathrm{Bar}}$) due to the effects of Barnett relaxation while J evolves slowly (on a timescale $t_{\mathrm{gas}} \gg t_{\mathrm{Bar}}$) due to gas damping. The disparity between t_{Bar} and t_{gas} means that, insofar as the motion of θ is concerned, we may regard J as a slowly-varying parameter. Thus, if we compute the measure of alignment by averaging the motion of θ over times much longer than t_{Bar} but much shorter than t_{gas}, the result will be $Q_X \approx Q_{X,\mathrm{TE}}$, the approximation becoming exact in the limit $\delta_B \to \infty$. We conclude that the asymptotic behavior of the solutions in Figures 3 and 4 occurs because, when $\delta_B \gg 1$, the distribution of θ is always close to the thermodynamic equilibrium distribution and the latter is independent of δ_B. Accordingly, we will refer to the case $\delta_B \gg 1$ as the "thermodynamic regime." Of course, $Q_{X,\mathrm{TE}}$ depends on the value of J, so the value of Q_X computed by the "short-time averaging" procedure described above will change quasi-statically as gas damping slowly changes the magnitude of the angular momentum. Thus, it is necessary to compute Q_X by averaging the motion over many gas damping times, as we have done for the calculations shown in Figures 3 and 4.

For real interstellar grains, the Barnett time is typically a few orders of magnitude shorter than the gas damping time [compare eqs. (2-6) and (4-20)] so the solutions of practical interest in future applications correspond to $\delta_{\mathrm{B}} \sim 10^2$–$10^3$. Although we are unable to compute solutions for δ_{B} values this large (because the Langevin equation becomes too stiff), Figures 3 and 4 suggest that the solutions are already close to the limit $\delta_{\mathrm{B}} \to \infty$ when $\delta_{\mathrm{B}} = 25$. Accordingly, values of Q_X computed for $\delta_{\mathrm{B}} = 25$ should be good approximations to the results of practical interest. We have calculated the measure of internal alignment as a function of a/b and $T_{\mathrm{s}}/T_{\mathrm{g}}$ for the case $\delta_{\mathrm{B}} = 25$; the resulting values of Q_X appear as the upper entries in Table 1. The entries for $T_{\mathrm{s}}/T_{\mathrm{g}} = 1$ provide an additional check on our calculations because the angular momentum distribution must be Maxwellian when all of the temperatures in the system are equal. We have plotted these entries as open circles in Figure 2; the rms error in Q_X is 6×10^{-3} for these points.

Because the distribution of θ is always close to the thermodynamic equilibrium distribution when δ_{B} is large, one should be able to approximate the values of Q_X in Table 1 by an expression of the form

$$Q_X \approx Q_{X,\mathrm{TE}}\left(a/b, T_{\mathrm{s}}, J_{\mathrm{eff}}\right), \tag{5-7}$$

where J_{eff} is some "effective J-value." We have computed the effective J value for each entry in Table 1 by interpreting expression (5-7) to be the *definition* of J_{eff}. In particular, each Q_X value in the table is given exactly by

$$Q_X = Q_{X,\mathrm{TE}}\left(a/b, T_{\mathrm{s}}, \alpha J_{\mathrm{Max}}\right), \tag{5-8}$$

where

$$J_{\mathrm{Max}} = \left\{ \left[1 + \frac{(a/b)^2}{2} \right] \left[1 + \frac{T_{\mathrm{s}}}{T_{\mathrm{g}}} \right] \right\}^{1/2} J_{\mathrm{therm}} \tag{5-9}$$

is the rms value of J for a Maxwellian angular momentum distribution and α is a dimensionless number. For each parameter combination, the lower entry in Table 1 gives the corresponding value of α. Figure 5 gives a graphical comparison between selected results from the table and the "$\alpha = 1$" approximation The agreement is generally good, although the $\alpha = 1$ approximation systematically underestimates Q_X for points with $Q_X \gtrsim 0.4$. This systematic discrepancy is likely due to errors in the numerical solutions associated with the singularity in $B_{\theta\theta}^{(\mathrm{Bar})}$ at $\theta = 0$ rather than the quality of the approximation. In view of the various uncertainties in grain properties, e.g., in the rotational temperature and rotational inertias, we regard the $\alpha = 1$ approximation as a reasonable method for estimating the degree of Barnett alignment in real grains when gas damping is the dominant external interaction.

6. Discussion

The study of internal relaxation presented in this paper includes the physics of Barnett relaxation and gas damping but does not include the Davis-Greenstein effect or Gold's mechanism. A quantitative study including the omitted processes is a challenging problem, which we address in subsequent papers in this series (Lazarian 1997a and Roberge & Lazarian 1997a for the Davis-Greenstein mechanism and Lazarian 1997b and Roberge & Lazarian 1997b for Gold's mechanism). Without going into details, we may summarize the qualitative effects of incomplete Barnett alignment on the DG and Gold mechanisms as follows. Because the DG and Gold mechanisms are both concerned with the alignment of $\boldsymbol{J}$ with respect to $\boldsymbol{B}$, the wobbling of a thermally-rotating grain about its angular momentum (which occurs when Barnett alignment is imperfect) will result in a decrease in its polarizing efficiency relative to the case for perfect Barnett alignment. According to calculations based on the assumption of perfect Barnett alignment (Lazarian 1994; Roberge, Hanany, & Messinger 1995), the efficiency of Gold alignment depends on the orientation dependence of a grain's cross section for collisions with streaming gas particles, in the sense that the alignment becomes more efficient as this difference increases. The precession of the grain axes about $\boldsymbol{J}$ which occurs when Barnett alignment is imperfect will generally reduce this difference. Consequently, we anticipate that imperfect Barnett alignment will reduce the efficiency of DG and Gold alignment for thermally-rotating grains. Conversely, the DG and Gold mechanisms affect the efficiency of Barnett alignment. Paramagnetic dissipation in the interstellar magnetic field will decrease the Barnett alignment by reducing the value of J. On the other hand, Gold's mechanism *increases* J and therefore increases the efficiency of Barnett alignment. Strangely, incomplete Barnett relaxation increases the efficiency of paramagnetic alignment if the grains rotate suprathermally by increasing the correlation in the directions of $\boldsymbol{J}$ before and after crossover events (Lazarian & Draine 1997).

7. Summary

The principal results of this paper are as follows:

1. We have presented an exact formulation of the physics of Barnett relaxation which includes the effects of thermal fluctuations in the grain material.

2. We have shown that the measure of internal alignment, Q_X, depends only on 3 dimensionless parameters in the approximation where the external torques on the grain are dominated by gas damping. These parameters are the grain axis ratio, a/b, the dust-to-gas temperature ratio, $T_\mathrm{s}/T_\mathrm{g}$, and a "Barnett damping parameter," δ_B.

3. We have calculated Q_X for a large number of parameter combinations.

4. We have obtained an analytic approximation that accurately reproduces our numerical solutions in the portion of parameter space relevant to studies of interstellar polarization.

5. The results of this paper confirm the conclusion of L94, that the alignment of $\boldsymbol{J}$ with respect to the axis of major inertia is incomplete for thermally-rotating grains. The incomplete alignment stems from thermal fluctuations within the grain material and becomes increasingly significant as the grain temperature rises. The rotation of a grain with $\boldsymbol{J}$ parallel to its major axis of inertia corresponds to the minimum kinetic energy for a given J value. Thermal fluctuations disorient $\boldsymbol{J}$ because they transfer energy from the grain solid material to the rotational degrees of freedom. In a steady state, energy is transferred back to the grain material at an equal rate by Barnett dissipation.

6. It is important to include incomplete Barnett alignment in studies which relate grain alignment theory to polarimetric data. The polarization of electromagnetic radiation is caused by the preferential alignment of the grain axes. At the same time, theoretical studies of alignment deal with anisotropies in the angular momentum distribution.[8] The distribution of $\boldsymbol{J}$ in grain coordinates is essential for relating observations to theory.

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[8]In fact, it is essential to know the orientation of the grain axes even at this stage, as the alignment (e.g., Gold alignment) depends on the axis orientation with respect to the magnetic field or to the gaseous flow.

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FIGURE CAPTIONS

Fig. 1 — Measure of alignment for an isolated grain with constant angular momentum, plotted versus the square root of the suprathermality parameter. Solid curve: exact solution from equation (3-4). Filled circles: approximate solutions obtained by integrating the Langevin equation. The rms error in the numerical solutions is 4×10^{-3}.

Fig. 2 — Measure of alignment in the Maxwellian regime plotted versus the grain axis ratio. Solid curve: exact solution from equation (5-5). Filled circles: numerical results for $\delta_B = 0$ and arbitrary T_s/T_g. Open circles: numerical results for $\delta_B = 25$ and $T_s/T_g = 1$. The rms error in the numerical solutions is 2×10^{-3} for the filled circles and 6×10^{-3} for the open circles.

Fig. 3 — Numerical calculations of the measure of alignment for a highly-flattened grain with $a/b = 0.1$ including the effects of gas damping and Barnett relaxation. Abcissa: Barnett damping parameter, δ_B [see eq. (5-3)]. Ordinate: Measure of internal alignment, Q_X. Solutions are presented for $T_s/T_g = 0.1$ (filled circles) and $T_s/T_g = 0.5$ (open circles). The dashed line is the exact solution in the Maxwellian regime and corresponds to $T_s/T_g = 1$.

Fig. 4 — Similar to Figure 3 but for a modestly flattened grain with $a/b = 0.5$.

Fig. 5 — Graphical comparison between the analytic approximation defined by expression (5-8) with $\alpha = 1$ (solid curves) and selected numerical results from Table 1 (symbols). Results are shown for $a/b = 0.1$ (top curve, filled circles), $a/b = 0.5$ (middle curve, open circles), and $a/b = 0.9$ (bottom curve, filled squares).

Table 1

Measure of Internal Alignment for $\delta_B = 25$

					$T_{\rm s}/T_{\rm g}$					
a/b	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.54	0.39	0.29	0.24	0.22	0.18	0.18	0.16	0.14	0.13
	0.85	0.95	0.96	0.98	1.01	0.98	1.03	1.00	0.98	0.97
0.3	0.49	0.35	0.25	0.22	0.18	0.17	0.15	0.14	0.14	0.13
	0.85	0.95	0.95	0.98	0.98	0.99	0.99	1.00	1.02	1.01
0.5	0.44	0.28	0.21	0.18	0.15	0.13	0.12	0.11	0.10	0.09
	0.90	0.97	0.98	1.04	1.02	1.00	1.00	1.01	1.00	0.96
0.7	0.30	0.17	0.13	0.11	0.09	0.08	0.07	0.07	0.06	0.06
	0.93	0.96	1.01	1.02	1.00	0.98	0.97	1.01	0.99	0.98
0.9	0.11	0.06	0.04	0.03	0.03	0.02	0.02	0.02	0.01	0.01
	0.98	0.97	0.97	0.97	0.94	0.93	0.88	0.84	0.82	0.81